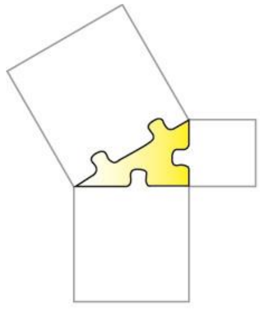
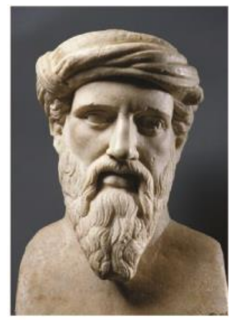
The pocket Pythagoras

Jacques Chaurette, November 2020
jchaurette@pythagoraspuzzle.com





PYTHAGORAS
(570-530 BCE)

The Bhaskara diagram (see figure 1) provides us with a simple way to prove Pythagoras's theorem without a single line of algebra; everyone can have their own pocket Pythagoras proof. These types of proofs are known as proofs by dissection or rearrangement.

The Bhaskara diagram

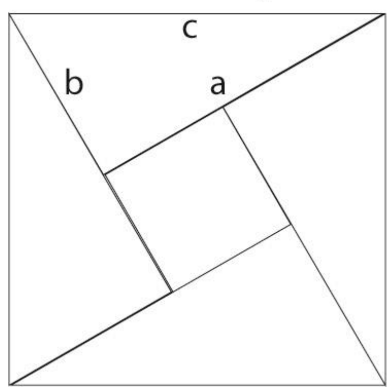


Figure 1

Solving puzzles is fun as long as there is a fair chance of succeeding and of course it can be sometimes frustrating. The reward for solving this puzzle is fundamental knowledge about our world.

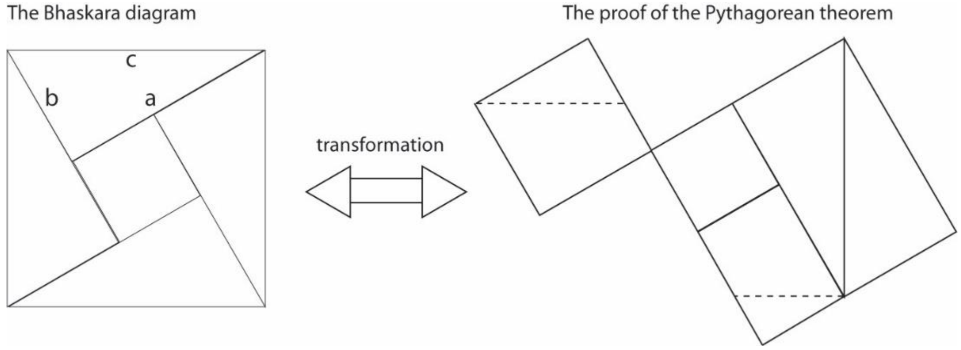


Figure 2

Pythagoras of Samos lived around 570-530 BCE, he was Greek and lived the latter part of his life in the town of Croton in the south of Italy. At this time, Greek colonies had spread all over the Mediterranean. Pythagoras is said to have travelled widely which is likely how he may have heard of the relationship between the sides of a right-angled triangle. He was a cult leader and believed that nature, the world and even the heavens were intimately connected to numbers.

A right-angle triangle is a triangle that has one right angle; according to Euclid's theorem 6.7 the sum of the angles of any triangle is equal to 180°, therefore there can only be one right angle in a triangle[4].

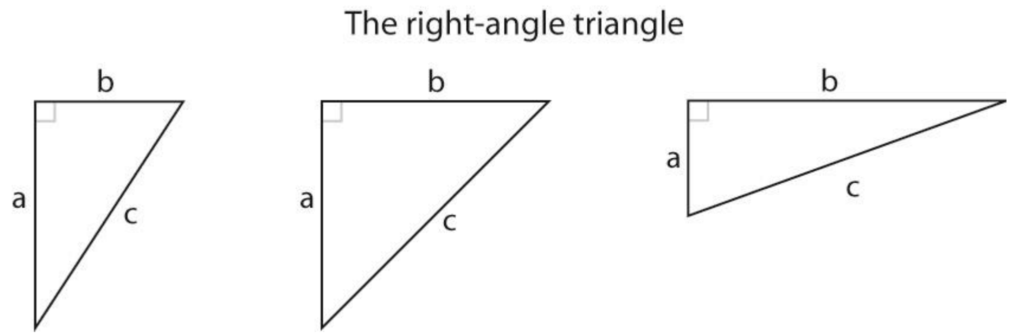


Figure 3 Different versions of a right-angled triangle

Pythagoras is famous for two major discoveries, the proof of the right-angle triangle theorem considered to be the foundation of mathematics[2] and the numerical relationship between musical tones; both these discoveries resonate with us today. Unfortunately, no records of Pythagoras' proof have been preserved. The Pythagoras theorem is also the source of trigonometry which was developed to assist in land survey and astronomical measurements[4].

In regards to music, he discovered that if you take a string under tension and reduce the length by ½, you get a tone that is one octave higher. If you change the length of the string to 2/3 of its original length you get what we now call the note G in the C major scale. This kind fractioning of the string length allows us to determine the remaining 4 notes of the major scale all of which are separated by precise musical intervals. This is the basis for all the notes in our major scale and harmony. This relationship between string length and tone produced by a taught string is now well understood. The note produced by a string depends on its length, the linear density of the material and the tension. When struck a string produces a distinct integer number of waves between the two fixed points; this has been studied by many scientists in particular E. Fourier who clearly described this phenomena and how different tones interact with each other.

This idea was later expanded to include the heavens as they were then known. The Earth was at the center and each planet including the Sun was on an individual sphere centered on the Earth. These crystal spheres were thought to vibrate with the frequencies that Pythagoras had discovered, hence the music of the spheres.

Getting back the right-angle triangle theorem, Pythagoras did not discover it and it was well known to the Babylonians 1500 years before his time. Pythagoras' great innovation was to prove the theorem. The Greeks were the first to concern themselves with proving such relationships. They understood that you can have many examples of something that is true, but that doesn't mean that it is true for all cases. They wanted bedrock knowledge, something that was true for all time and on which you can build further knowledge with the assurance that the foundation is rock solid. This is exemplified by Euclid's "The elements" c 300 BCE, a compilation of all the knowledge of the time on geometry and other matters, a foundational document for 1000 years to come. In time the right-angle triangle theorem became known as the Pythagoras theorem.

I came upon Bhaskara's diagram in Figure 1 in a book called The Pythagorean Proposition by Elisha Scott Loomis that has over 370 proofs of the Pythagoras theorem. It was never my intention to study all the proofs but I was curious why so many proofs had been produced and maybe some of these proofs would contain some gems as is often the case in a good proof. Most of them are quite lengthy involving many construction lines. It is not surprising that they are seldom taught in our school system, and often the students are provided with the result only $c^2 = a^2 + b^2$. Hidden away within this book is the image in Figure 1. It is credited to the Hindu mathematician Bhaskara ("The Learned" 1114-1185 CE)[1]. The only caption that comes with the image is "This image is all you need to know to prove the Pythagoras theorem."

The letters for the lengths of the sides of the right-angle triangle were added. There are no markings on the figure in the book. Of course, a statement that this is "all you need to know…" is very intriguing and provoked me into discovering its secrets. It turns out that what the Indian gentleman said is true and there is more than one secret.

We can arrive at the proof of the Pythagorean theorem by manipulating the different shapes in the Bhaskara diagram.

To know this important theorem and its proof is empowering for anyone, it is one of those secrets of the universe that we can hold and treasure when everything else is in doubt.

One consequence of the Pythagorean theorem is a result that surprised and shocked Pythagoras and his followers. In fact, it rocked their world to the core and disturbed the number system in a radical fashion. If we draw a square of one unit per side and draw the diagonal, we get a right-angle triangle with two equal sides of a unit length of 1, then by Pythagoras's theorem the hypothenuse will equal to the square root of 2.

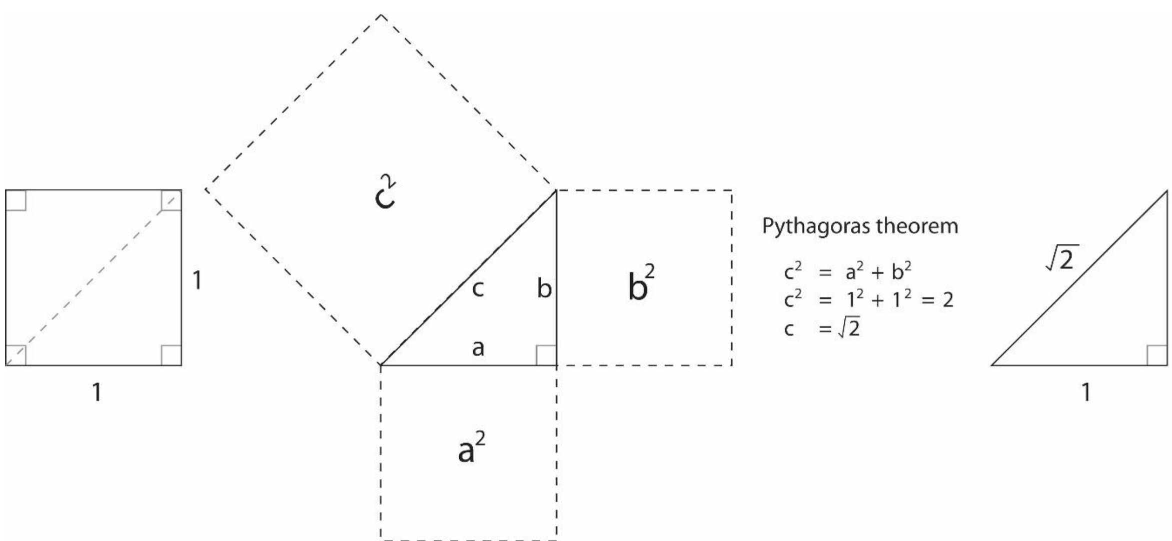


Figure 4 Irrational numbers

What is remarkable about $\sqrt{2}$ is that there is no fraction or ratio that you could come up with that would express that number exactly. The value of $\sqrt{2}$ is 1.4142135624… The trailing dots means that these numbers go on forever and moreover the digits are random. This is different from any fraction or ratio of whole numbers; these have a finite number of digits or a repeating pattern. For example, 1/4 = 0.25, 2/3 = 0.6666… or sometimes written as $0.\overline{6}$ where the horizontal bar indicates a repeating number. For the Pythagoreans this meant that you could not measure a line segment of length $\sqrt{2}$ and it was said that a length of this type was incommensurable. Incommensurable means that there is no scale or line no matter how finely divided that could be used to measure a number such as $\sqrt{2}$. The early Greeks came up with a simple and elegant proof of this. We now understand these numbers very well and they are called *irrational* numbers, not because they are crazy in some way but because they do not have that same quality of normality of integer numbers and ratios of whole numbers. We should remember that negative numbers were also viewed with deep suspicion prior to the 1500's.

This was very shocking to the Pythagoreans and it wasn't until Stevin in 1585 that we came to terms with the nature of these numbers and accepted irrational numbers as any other number by expanding our view of the number system. Today we use an approximate value for the $\sqrt{2}$ that for practical purposes we can make as precise as we want or as needed.

However, there are mathematicians who debate the nature of irrational numbers, one of them is Prof. Wildberger who has a very interesting Youtube channel on the history and foundations of mathematics[3].

This is one reason why the Greeks preferred working with surfaces or areas instead of line lengths.

For example, the length of the hypothenuse of the triangle in Figure 4a is sqrt(2^2 + 1^2) = $\sqrt{5}$. Not an easy number to calculate, but we can calculate the area of the square that has a common side with the hypothenuse of the triangle[3].

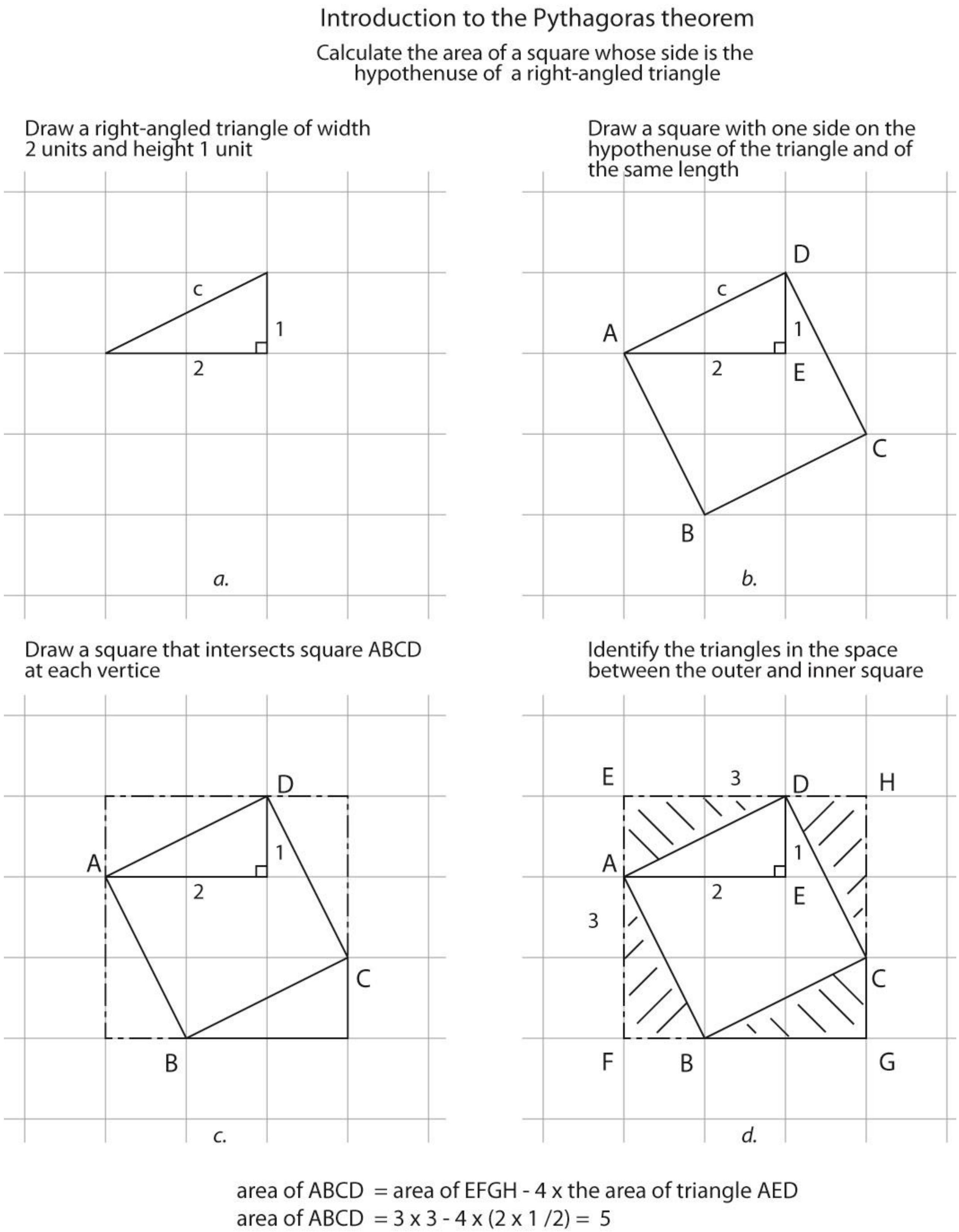


Figure 5 Introduction to the Pythagoras theorem.

Let's take a look at the traditional proof of the Pythagorean theorem. The Ancient Greeks preferred to deal with whole or natural numbers which is easily done by considering areas instead of line lengths.

The traditional proof of the Pythagoras theorem is done by moving surfaces around in such a way as to create 2 squares one which has area A2 corresponding to the square formed by the small side (vertical) of triangle 1, and another of area A1 corresponding to the square formed by the horizontal side of triangle 1. It becomes obvious that the sum of these 2 areas is equal to the area A3 corresponding to the square formed by the hypothenuse of triangle 1 which is the proof of Pythagoras' theorem.

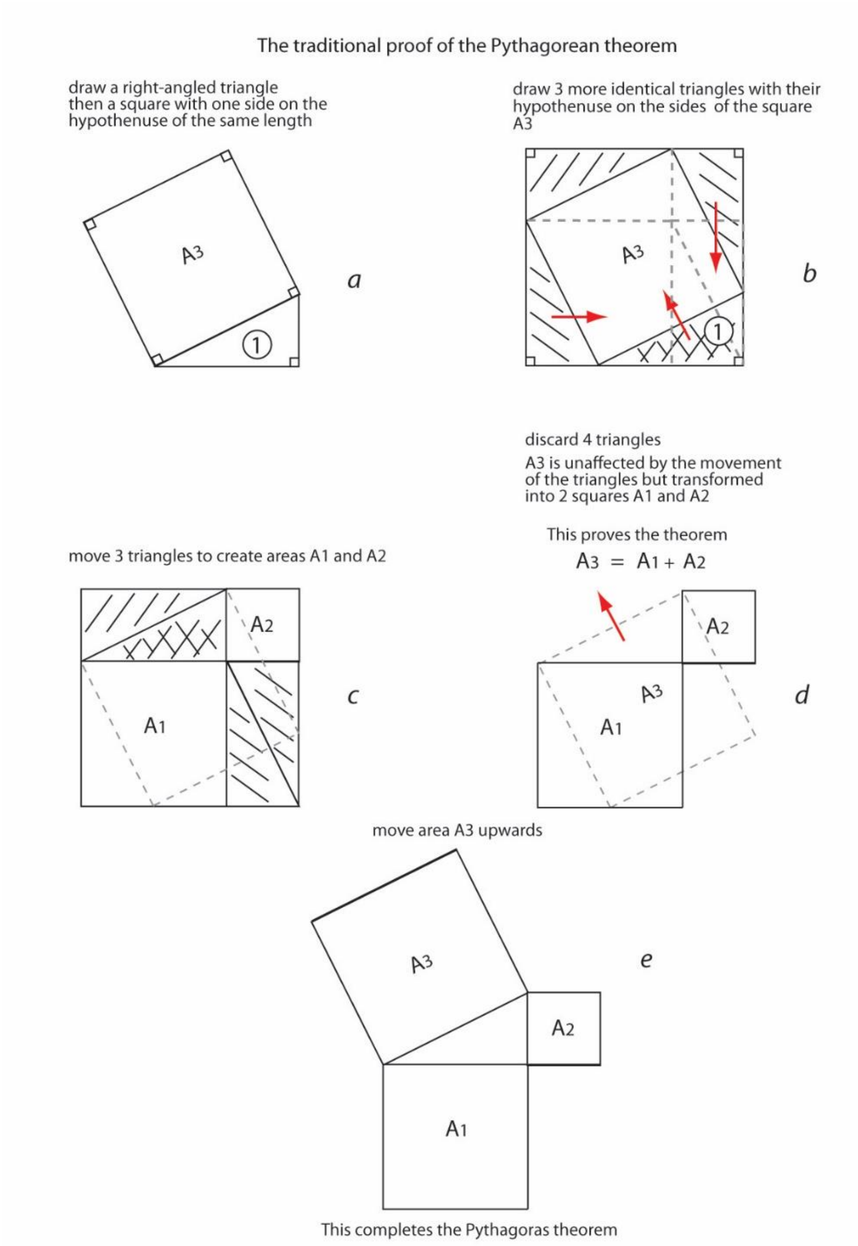


Figure 6 The traditional proof of the Pythagorean theorem.

The Bhaskara diagram (see figure 1) is a slightly different construction than the traditional proof diagram. As one might expect there is a relationship between the two.

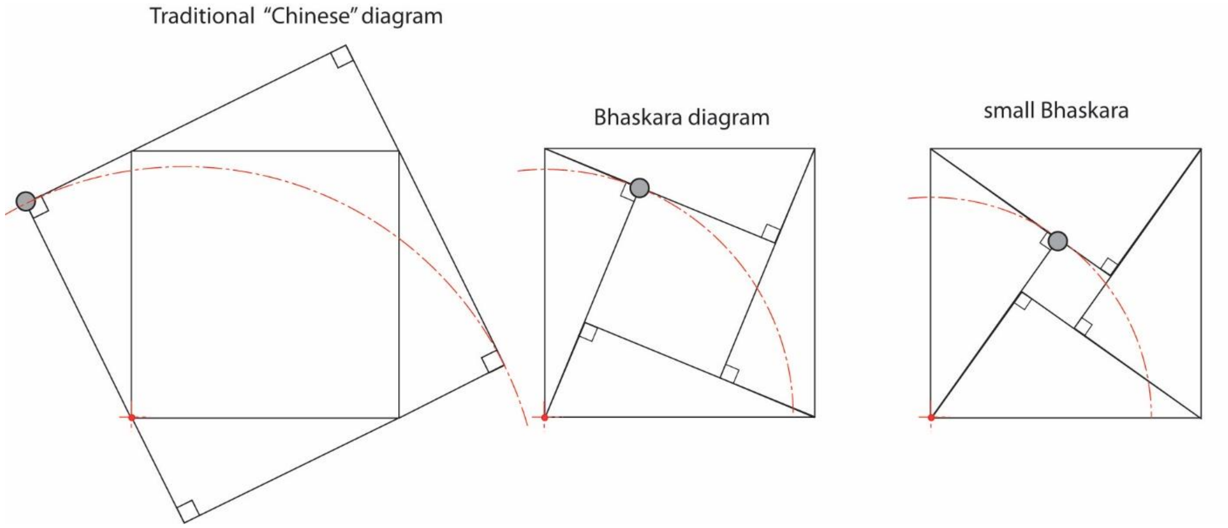


Figure 7 The transformation of the Pythagoras diagram into the Bhaskara diagram.

The next figure shows the proof of the Pythagorean theorem using the Bhaskara diagram. One advantage of Bhaskara is that all the pieces that make up the diagram are used; as opposed to the traditional diagram where one surface A3 is transformed into 2 square surfaces A1 and A2. It makes the Bhaskara diagram more suitable to produce as a physical object with pieces to manipulate.

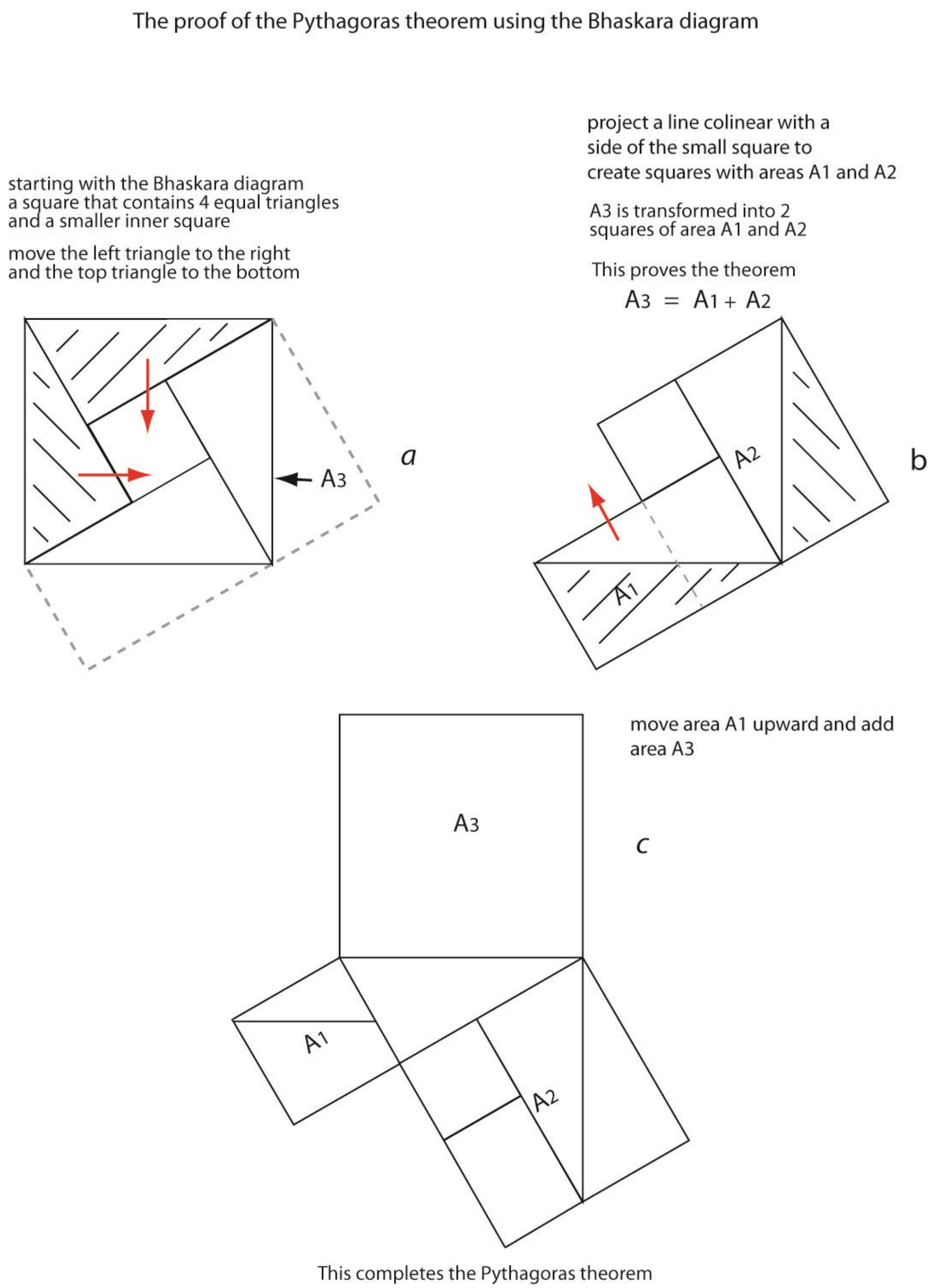


Figure 8 The proof of the Pythagorean theorem using the Bhaskara diagram.

Interestingly the traditional proof diagram is really a special case of the Bhaskara diagram. The size of the center square of the Bhaskara diagram can be increased from zero to a size that exceeds the surrounding square creating the traditional proof diagram. Alternatively, we can work our way from a simple triangle to the traditional diagram to the Bhaskara diagram as shown in the next Figure. Also, in position f of the figure there is a short proof of why the triangles always have a right-angle as the center square goes from zero to the large size of the traditional diagram.

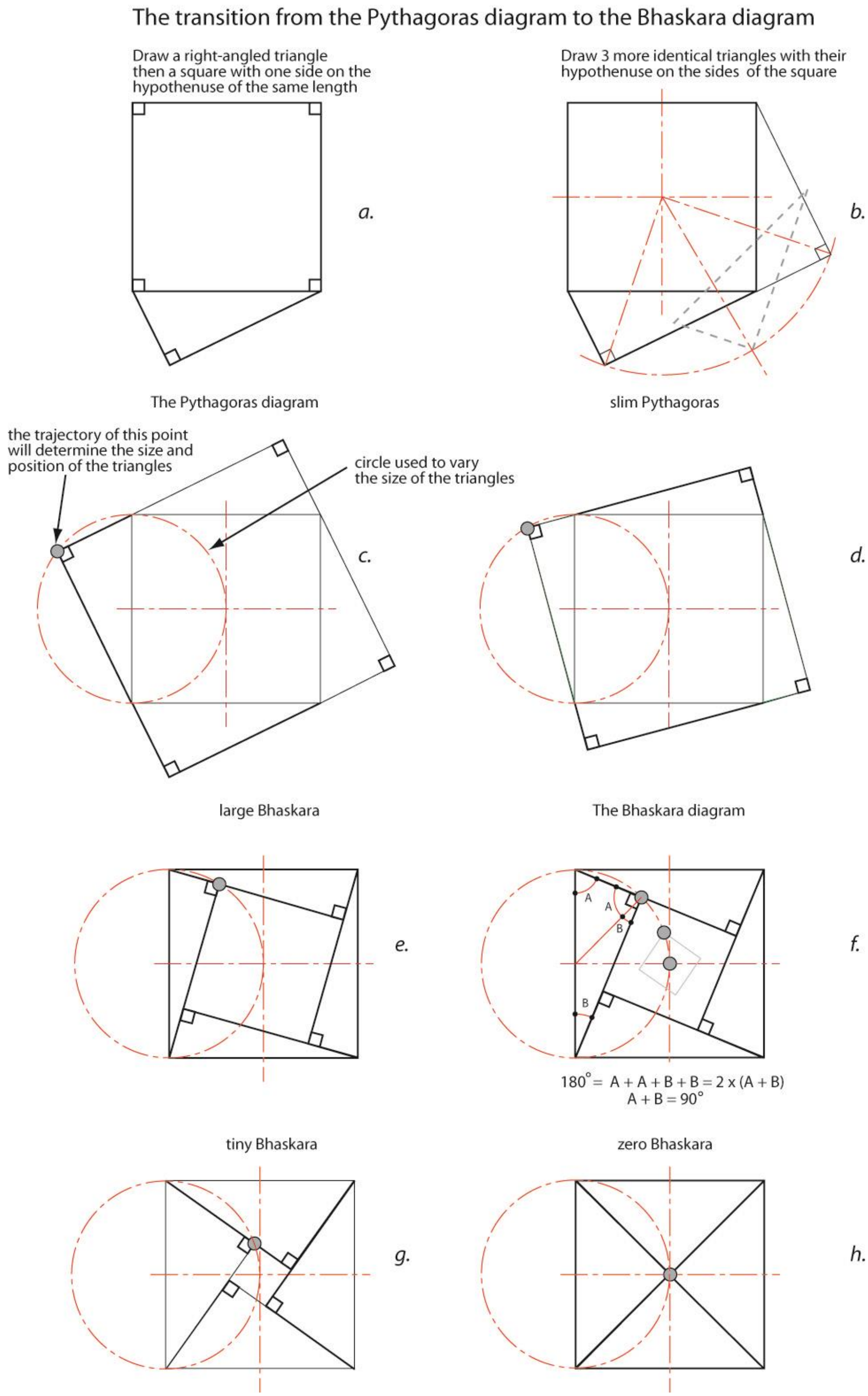


Figure 9 The traditional Pythagoras diagram transitioning to the Bhaskara diagram.

There is one last surprise that Bhaskara offers us, we can keep the transformation going in the previous figure and cycle back to the traditional proof diagram.

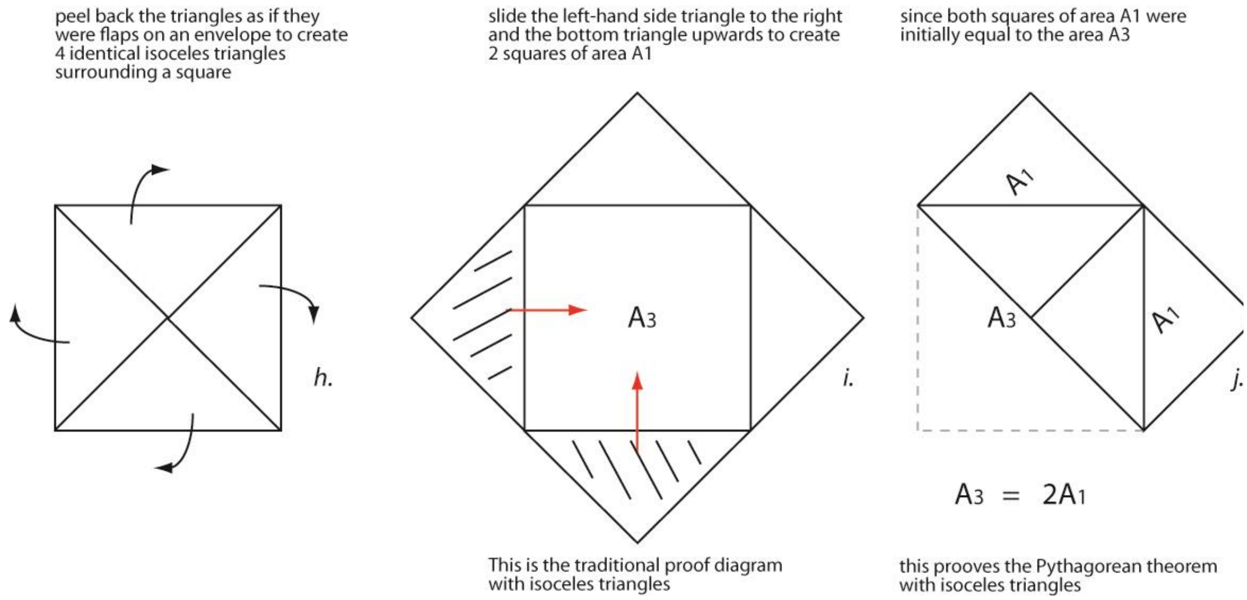


Figure 10 The last transition of the Bhaskara diagram bring us back to the Pythagoras diagram.

I hope you enjoy doing the Pythagoras puzzle that Bhaskara left us to ponder over as much as I did. It's confusing at first like any good puzzle and the solution is surprising and gratifying.

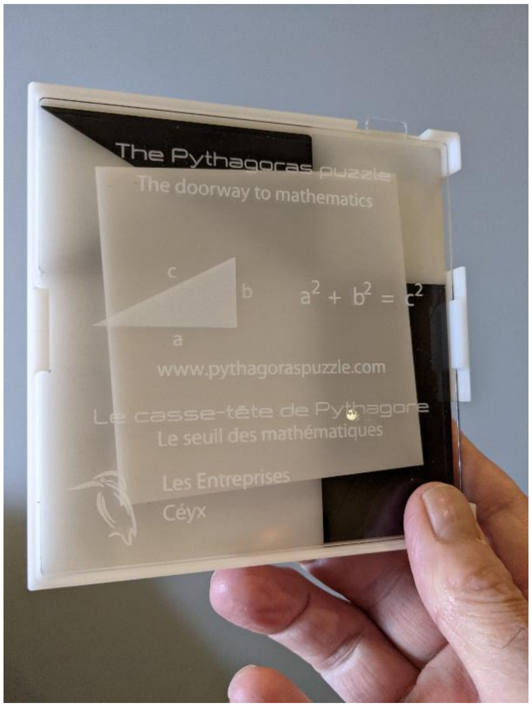
Ref. 1 The Pythagorean Theorem, Eli Maor

Ref. 2 The Ascent of Man, Jacob Bronowsky

Ref. 3 Math Foundations, Prof. Norman Wildberger, https://www.youtube.com/watch?v=REeaT2mWj6Y&list=PLIljB45xT85DpiADQOPth56AVC48SrPLc&index=2&t=0s

Ref. 4 Euclid's "Elements"

The author has produced a physical version of these demonstrations allowing for the movement of the surfaces proving Pythagoras’ theorem.



This prototype is in the final stage and a production version will be available soon. Here is a short video showing its use:

https://www.pythagoraspuzzle.com/the%20puzzle.mp4